Abstract

In this paper we present a generalization of the Goraychev–Chaplygin integrable case on a bundle of Poisson brackets, and on Sokolov terms in his new integrable case of Kirchhoff equations. We also present a new analogous integrable case for the quaternion form of rigid body dynamics' equations. This form of equations is recently developed and we can use it for the description of rigid body motions in specific force fields, and for the study of different problems of quantum mechanics. In addition we present new invariant relations in the considered problems.

GENERALIZATION OF THE GORAYCHEV–CHAPLYGIN CASE

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In this paper we present a generalization of the Goraychev–Chaplygin integrable case on a bundle of Poisson brackets including (co)algebras $so(4)$ and $so(3,1)$, and on the quaternion form of Euler–Poisson equations. Note that the generalization on the bundle is connected with the introduction of variables on the bundle analogous to Andoyer–Deprit variables. These variables are separating for all members of the bundle. We also obtain the $L-A$-pair of the generalized Goraychev–Chaplygin case of quaternion equations. In a particular case this pair is reduced to the $L-A$-pair of the classical Goraychev–Chaplygin case.

Let us consider a generalization of the classical Goraychev–Chaplygin case of Euler–Poisson equations

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \frac{\partial H}{\partial \boldsymbol{M}}$$

for the zero value of the area integral $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$. We add a gyrostatic moment and singular term to the Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) + \lambda M_3 + \mu\gamma_1 + \frac{1}{2}\frac{a}{\gamma_3^2}, \tag{1}$$

where $\lambda, \mu, a = \mathrm{const}$. The additional integral is of order three with respect to the moments

$$F = \left(M_3 + \frac{\lambda}{2}\right)\left(M_1^2 + M_2^2 + \frac{a}{\gamma_3^2}\right) - \mu M_1\gamma_3.$$

D. N. Goraychev himself in the paper [7] showed the generalization (1) on zero value surface of gyrostatic moment $\lambda = 0$ (for $\lambda \neq 0$, $a = 0$ it was

shown by L. N. Sretensky [9]). The complete form of generalization (1) was considered by I. V. Komarov and V. B. Kuznetsov [8],. They also presented some quantum mechanical interpretation of the singular term.

The Poisson bracket of variables $\boldsymbol{M}, \boldsymbol{\gamma}$ is defined by algebra $e(3)$ and has the following form

$$\{M_i, M_j\} = -\varepsilon_{ijk}M_k, \quad \{M_i, \gamma_j\} = -\varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0. \tag{2}$$

Let us present the generalization of the Goraychev–Chaplygin case on the bundle of brackets of the form

$$\{M_i, M_j\} = -\varepsilon_{ijk}M_k, \quad \{M_i, \gamma_j\} = -\varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = -x\varepsilon_{ijk}M_k. \tag{3}$$

At $x = 1$ these commutation relations correspond to the algebra $so(4)$.

We can present the Casimir functions of bracket (3) in the form

$$F_2 = \boldsymbol{\gamma}^2 + x\boldsymbol{M}^2, \quad F_1 = (\boldsymbol{M}, \boldsymbol{\gamma}). \tag{4}$$

Now let us construct the variables on bracket (3) analogous to the Andoyer–Deprit variables [1].

Suppose the component M_3 is equal to the momentum

$$L = M_3. \tag{5}$$

The variable l canonically conjugate to the variable L ($\{l,\, L\} = 1$) on subalgebra $so(3)$ with the generators M_1, M_2, M_3 is constructed by integration of the Hamiltonian flow with Hamiltonian function $\mathcal{H} = L$

$$\begin{gathered}\frac{dM_1}{dl} = \{M_1,\, L\} = M_2, \quad \frac{dM_2}{dl} = \{M_2,\, L\} = -M_1, \\ \frac{dM_3}{dl} = \{M_3,\, L\} = 0.\end{gathered} \tag{6}$$

Hence using the commutation relation $\{M_2,\, M_2\} = -M_3$ we obtain

$$M_1 = \sqrt{G^2 - L^2}\sin l, \qquad M_2 = \sqrt{G^2 - L^2}\cos l, \tag{7}$$

where $G^2 = M_1^2 + M_2^2 + M_3^2$ is the Casimir function of subalgebra $so(3)$.

Suppose G is the second momentum and construct the canonically conjugate variable g. We chose $H = G$ as a new Hamiltonian, and the corresponding flow on the whole bundle $\mathcal{L}_x$ has the form

$$\frac{d\boldsymbol{M}}{dg} = 0, \qquad \frac{d\boldsymbol{\gamma}}{dg} = \frac{1}{G}\boldsymbol{\gamma} \times \boldsymbol{M}, \tag{8}$$

where g is the variable canonically conjugate to G.

According to (8), $\boldsymbol{M}$ does not depend on g, and using equations (8) and Casimir functions (4) we obtain for $\boldsymbol{\gamma}$ the equations

$$\begin{gathered}\boldsymbol{\gamma} = \frac{H}{G^2}\boldsymbol{M} + \frac{\alpha}{G}(\boldsymbol{M}\times\boldsymbol{e}_3\sin g + G\boldsymbol{M}\times(\boldsymbol{M}\times\boldsymbol{e}_3)\cos g),\\ \alpha^2 = \frac{c_2 - xG^2 - \frac{H^2}{G^2}}{G^2 - L^2}, \qquad \boldsymbol{e}_3 = (0,\,0,\,1),\end{gathered} \tag{9}$$

where $c_2 = x\boldsymbol{M}^2 + \boldsymbol{\gamma}^2$, and H is the traditional notation of the area integral $H = (\boldsymbol{M}, \boldsymbol{\gamma}) = c_1$.

Thus (5), (7), (9) define the symplectic variables on the whole bundle $\mathcal{L}_x$ corresponding at $x = 0$, $c = 1$ to the well-known Andoyer–Deprit variables in rigid body dynamics.

Using (5), (9) we obtain the generalization of the particular Goraychev–Chaplygin integrable case on the bundle $\mathcal{L}_x$. We chose the Hamiltonian in the form

$$\mathcal{H} = \frac{1}{2}(G^2 + 3L^2) + \lambda L + a(\cos l\cos g + \frac{L}{G}\sin l\sin g), \tag{10}$$

where a, λ are constants.

In (10) we add the linear with respect to L term. It is interpreted on algebra $e(3)$ as the component of the gyrostatic moment [9].

We can separate variables in system (10). Indeed, let us apply the canonical change of variables

$$L = p_1 + p_2, \quad G = p_1 - p_2, \qquad q_1 = l + g, \quad q_2 = l - g. \tag{11}$$

Now Hamiltonian (10) can be presented in the form

$$\mathcal{H} = \frac{1}{2}\frac{p_1^3 - p_2^3}{p_1 - p_2} - \lambda\frac{p_1^2 - p_2^2}{p_1 - p_2} + \frac{a}{p_1 - p_2}(p_1\sin q_1 + p_2\sin q_2). \tag{12}$$

Using (5), (7), (9) we present Hamiltonian (10) as a function of variables $\boldsymbol{M}$, $\boldsymbol{\gamma}$ on the zero surface of area integral $(\boldsymbol{M},\,\boldsymbol{\gamma}) = H = 0$

$$\mathcal{H} = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) + \lambda M_3 + \mu\frac{\gamma_1}{|\boldsymbol{\gamma}|}. \tag{13}$$

The additional integral in this case has the form

$$F = \Big(M_3 + \frac{\lambda}{2}\Big)(M_1^2 + M_2^2) - \mu M_1 \frac{\gamma_3}{|\boldsymbol{\gamma}|}. \tag{14}$$

On algebra $e(3)$ we have $|\boldsymbol{\gamma}| = 1$ and obtain the classical Goraychev–Chaplygin integrable case. In the classical case this method of variables' separation was suggested by *V. B. Kozlov* [2].

Singular term (1) is generalized on the bundle in the following way:

$$H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) + \lambda M_3 + \mu\frac{\gamma_1}{|\boldsymbol{\gamma}|} + \frac{1}{2}\frac{a\boldsymbol{\gamma}^2}{\gamma_3^2},$$
$$F = \Big(M_3 + \frac{\lambda}{2}\Big)\Big(M_1^2 + M_2^2 + a\frac{\gamma_2}{\gamma_3^2}\Big) - \mu M_1\gamma_3,$$

although variables' change (11) in this case doesn't produce the separation of variables (at least we don't know such separation).

Recently Sokolov and Tsyganov present a new generalization of particular integrable case (1) on bracket (2) with the Hamiltonian containing the quadratic cross terms with respect to $\boldsymbol{M}, \boldsymbol{\gamma}$.

The most general form of the integrable family in this case is presented as the following Hamiltonian

$$\begin{aligned} H = \frac{1}{2}\Big(M_1^2 + M_2^2 + 4M_3^2 + \frac{\varepsilon}{\gamma_3^2}\Big) + \lambda M_3 + \mu_1\gamma_1 + \mu_2\gamma_2 + \\ +a_1(2M_3\gamma_1 - M_1\gamma_3) + a_2(2M_3\gamma_2 - M_2\gamma_3). \end{aligned} \tag{15}$$

The additional integral is

$$F = \Big(M_3 + a_1\gamma_1 + a_2\gamma_2 + \frac{\lambda}{2}\Big)\Big(M_1^2 + M_2^2 + \frac{\varepsilon}{\gamma_3^2}\Big) - (\mu_1 M_1 + \mu_2 M_2)\gamma_3. \tag{16}$$

At $\varepsilon = 0$ we can separate variables in Hamiltonian (15) using the linear combination of Andoyer–Deprit variables analogous to (11). Indeed, we can show that

$$\begin{aligned} H &= \frac{1}{p_1 - p_2}\big(f(p_1, q_1) - g(p_2, q_2)\big), \\ f &= 2a_1p_1^3 + 2\Big(\frac{\lambda}{2} + a_1\sin q_1 + a_2\cos q_2\Big)p_1^2 + (\mu_1\sin q_1 + \mu_2\cos q_1)p_1, \\ g &= 2a_1p_2^3 + 2\Big(\frac{\lambda}{2} - a_1\sin q_2 - a_2\cos q_2\Big)p_2^2 - (\mu_1\sin q_2 + \mu_2\cos q_2)p_2. \end{aligned} \tag{17}$$

As we note above, we don't know such separation at $\varepsilon \neq 0$.

If the gravity field is absent ($\mu_1 = \mu_2 = 0$), then the integral is presented as a product of factors $F = k_1 k_2$, where

$$k_1 = a_0 M_3 + a_1\gamma_1 + a_2\gamma_2 + \frac{\lambda}{2}, \quad k_2 = M_1^2 + M_2^2 + \frac{\varepsilon}{\gamma_3^2}. \tag{18}$$

We can easily show that the equations of motion for k_1, k_2 have the form

$$\dot{k}_1 = 2(a_1\gamma_2 - a_2\gamma_1)k_1, \quad \dot{k}_2\Big|_{(\boldsymbol{M},\boldsymbol{\gamma})=0} = -2(a_1\gamma_2 - a_2\gamma_1)k_2.$$

Thus the equations $k_1 = 0$ and $k_2 = 0$ define the invariant relations of system (15) at $\mu_1 = \mu_2 = 0$ (and $k_2 = 0$ is the particular invariant relation on zero level of $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$, and $k_1 = 0$ is general invariant relation). It seems that, invariant relations (18) and the additional integral in the form $F = k_1 k_2$ where introduced by authors.

Using the observation from the first section we can generalize integrable case (15) and invariant relations on bundle of brackets (3). We shall make the following substitution in Hamiltonian (15), integral (16), and invariant relations (18)

$$\gamma_i \to \frac{\gamma_i}{|\boldsymbol{\gamma}|}, \quad |\boldsymbol{\gamma}| = \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}.$$

In this case at $\varepsilon = 0$ we can also separate variables using the analog of Andoyer–Deprit variables and substitution (11).

We should note that unlike the generalization of Kovalevskaya case [1], where the Hamiltonian and integral explicitly depend on the parameter of bundle x, in the generalization of Goraychev–Chaplygin case the integrals don't depend on the parameter of bundle.

Let us consider another possibly not so natural case of motion equations of rigid body with a potential linear with respect to Rodrig–Hamilton parameters and not with respect to direction cosines [1]

$$H = \frac{1}{2}(\mathbf{A}\boldsymbol{M}, \boldsymbol{M}) + \sum_{i=0}^{3} r_i \lambda_i, \quad r_i = \text{const}, \tag{19}$$

The equations of motion in this case are

$$
\begin{gathered}
\dot{\boldsymbol{M}} = \boldsymbol{M} \times \frac{\partial H}{\partial \boldsymbol{M}} + \frac{1}{2}\boldsymbol{\lambda} \times \frac{\partial H}{\partial \boldsymbol{\lambda}} + \frac{1}{2}\frac{\partial H}{\partial \lambda_0}\boldsymbol{\lambda} - \frac{1}{2}\lambda_0 \frac{\partial H}{\partial \boldsymbol{\lambda}}, \\
\dot{\lambda}_0 = -\frac{1}{2}\Big(\boldsymbol{\lambda}, \frac{\partial H}{\partial \boldsymbol{M}}\Big), \quad \dot{\boldsymbol{\lambda}} = \frac{1}{2}\boldsymbol{\lambda} \times \frac{\partial H}{\partial \boldsymbol{M}} + \frac{1}{2}\lambda_0 \frac{\partial H}{\partial \boldsymbol{M}},
\end{gathered}
\tag{20}
$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$.

These equations are Hamiltonian one with the Poisson bracket

$$
\begin{gathered}
\{M_i, M_j\} = -\varepsilon_{ijk} M_k, \quad \{M_i, \lambda_0\} = \frac{1}{2}\lambda_i, \\
\{M_i, \lambda_i\} = -\frac{1}{2}(\varepsilon_{ijk}\lambda_k + \delta_{ij}\lambda_0), \quad i, j, k = 1, 2, 3,
\end{gathered}
\tag{21}
$$

corresponding to the Lie algebra $so(3) \otimes_s \mathbb{R}^4$. The relations between quaternions and Euler angles have the form

$$
\begin{gathered}
\lambda_0 = \cos\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}, \quad \lambda_1 = \sin\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}, \\
\lambda_2 = \sin\frac{\theta}{2}\sin\frac{\psi-\varphi}{2}, \quad \lambda_3 = \cos\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}.
\end{gathered}
$$

In the classical mechanics such potential aren't presented because its dependance on the body position is nonunique (more exactly it is double-valued function). To justify the study of such equations we can refer to the problems of quantum mechanics and point masses dynamics in curved space S^3, and on some formal technics of $\mathbf{L}-\mathbf{A}$-pair construction [1, 5]. It turns out that after the reduction of order of system (19) we obtain the general Euler–Poisson equations with additional terms having different physical interpretations [1].

As an interesting feature of system (19) we note that using linear with respect to λ_i transformations we can reduce the general form of the potential

$$
V = \sum_{i=0}^{3} r_i \lambda_i \tag{22}
$$

to the form

$$
V = r_0 \lambda_0. \tag{23}
$$

Indeed, the linear transformations of quaternion space λ_i (preserving commutation relations and quaternion norm) of the form

$$\begin{gathered}\widetilde{\lambda}_0 = R^{-1}(r_0\lambda_0 + r_1\lambda_1 + r_2\lambda_2 + r_3\lambda_3),\\ \widetilde{\lambda}_1 = R^{-1}(r_0\lambda_1 - r_1\lambda_0 - r_2\lambda_3 + r_3\lambda_2),\\ \widetilde{\lambda}_2 = R^{-1}(r_0\lambda_2 + r_1\lambda_3 - r_2\lambda_0 - r_3\lambda_1),\\ \widetilde{\lambda}_3 = R^{-1}(r_0\lambda_3 - r_1\lambda_2 + r_2\lambda_1 - r_3\lambda_0),\\ R^2 = r_0^2 + r_1^2 + r_2^2 + r_3^2\end{gathered} \tag{24}$$

reduce potential (22) to form (23). The existence of such linear transformation is the remarkable property of quaternion variables and bracket (21), the analogous transformation does not exists for the brackets of algebras $e(3)$ and $so(4)$.

In general dynamically unsymmetrical case $a_1 \neq a_2 \neq a_3 \neq a_1$ system (19) seems to be nonintegrable and none of two necessary additional integrals exists. However, this was not proved, and the proof by various reasons is not simple. Note that even the application of the Kovalevskaya method for system (19) is not quite analogous to the classical Euler–Poisson problem. Generally speaking even in the Euler–Poinsot case the solution branches on the complex plane of time (with the exponent 1/2).

At $a_1 = a_2$ the linear integral always exists

$$\begin{gathered}F_1 = M_3(r_0^2 + r_1^2 + r_2^2 + r_3^2) + N_3(r_1^2 + r_2^2 - r_0^2 - r_3^2) +\\ + 2N_2(r_1 r_0 - r_3 r_2) - 2N_1(r_1 r_2 - r_0 r_3),\end{gathered} \tag{25}$$

where N_i are projections of kinetic moment on the fixed axes. For potential (23) this integral has the natural form

$$F_1 = M_3 - N_3. \tag{26}$$

It turns out that (linear) integral (26) corresponds to the cyclic variable $\varphi + \psi$ [1]. Rauss reduction with respect to this cyclic variable results in Hamiltonian system on algebra $e(3)$ on zero surface of area integral $(\boldsymbol{M}, \boldsymbol{\gamma}) = 0$ with the Hamiltonian

$$H = \frac{1}{2}(M_1^2 + M_2^2 + a_3 M_3^2) + c(a_3 - 1)M_3 + r_0\gamma_2 + \frac{1}{2}\frac{c^2}{\gamma_3^2}, \tag{27}$$

where c is a constant value of integral (26). Hamiltonian (27) corresponds to the addition of a gyrostatic term linear with respect to $\boldsymbol{M}$ and singular term into the general Euler–Poisson equations. The physical meaning of singular term is discussed in [1].

Integrable cases of system (19) (equivalent to integrable cases of system (27)) are presented in [1]. Here we just present the generalization of GoraychevĈhaplygin case.

Hamiltonian and additional integral are

$$\begin{aligned} H &= \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) + r_0\lambda_0, \\ F_2 &= M_3(M_1^2 + M_2^2) + r_0(M_2\lambda_1 - M_1\lambda_2). \end{aligned} \tag{28}$$

Integral F_2 commute with integral F_1. Under the reduction to system (27) this case becomes a member of generalized family (1).

Remark 1. *Adding a constant gyrostatic moment along the axis of dynamical symmetry in* (28) *we obtain an integrable case corresponding to the generalized Sretensky case in Euler–Poisson equations. Additional integrals can be easily obtained with the help of the lifting procedure described in [1].*

Note that the indicated "Goraychev–Chaplygin case" for the quaternion Euler–Poisson equations is *the general integrable case*! Thus we can use it for some algebraic constructions ($\mathbf{L} - \mathbf{A}$-pair constructions and all that) described below.

Lax representation of systems on algebra $su(2,1)$. We start from some formal method of **L-A** pair construction based on existence of two compatible Poisson brackets for the same system of Hamiltonian equations on Lie algebra; in this case the system is called bihamiltonian. The detailed exposition of this method and its applications to various problems of rigid body dynamics are presented in [5, 4].

Let's consider space $\mathcal{L}$ of complex matrixes 3×3 with the basis

$$\begin{aligned}
&\mathbf{M}_1 = \left(\begin{array}{cc|c} 0 & \frac{1}{2}i & \multirow{2}{*}{0} \\ \frac{1}{2}i & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \qquad
&&\mathbf{M}_2 = \left(\begin{array}{cc|c} 0 & \frac{1}{2} & \multirow{2}{*}{0} \\ -\frac{1}{2} & 0 & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \\
&\mathbf{M}_3 = \left(\begin{array}{cc|c} -\frac{1}{2}i & 0 & \multirow{2}{*}{0} \\ 0 & \frac{1}{2}i & \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right), \qquad
&&\mathbf{M}_4 = \left(\begin{array}{cc|c} -\frac{1}{6}i & 0 & \multirow{2}{*}{0} \\ 0 & -\frac{1}{6}i & \\ \hline \multicolumn{2}{c|}{0} & \frac{1}{3}i \end{array}\right), \\
&\mathbf{P}_1 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \frac{1}{2} \\ & & 0 \\ \hline -\frac{1}{2}x & 0 & 0 \end{array}\right), \qquad
&&\mathbf{P}_2 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & -\frac{1}{2}i \\ & & 0 \\ \hline -\frac{1}{2}xi & 0 & 0 \end{array}\right), \\
&\mathbf{P}_3 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 \\ & & \frac{1}{2} \\ \hline \frac{1}{2}xi & 0 & 0 \end{array}\right), \qquad
&&\mathbf{P}_4 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 0 \\ & & -\frac{1}{2}i \\ \hline 0 & \frac{1}{2}xi & 0 \end{array}\right).
\end{aligned} \tag{29}$$

With respect to the standard matrix commutator $[\cdot\ \cdot]$ such matrices generate a semi-simple algebra with Cartan decomposition $\mathcal{L} = H + V$, where subalgebra $H = su(2) \oplus su(1)$ is generated by matrices $\mathbf{M}_i$, and $V = \mathbb{C}^2$ is generated by matrices $\mathbf{P}_i$. Here x is a parameter determining some bundle of algebras linearly dependent on x. For $x > 0$ these algebras are isomorphic to algebra $su(3)$, for $x < 0$ they are isomorphic to algebra $su(2,1)$, at $x = 0$ they are isomorphic to the semidirect sum $(so(2) \oplus su(1)) \oplus_s \mathbb{C}^2$.

Using the semi-simplicity we can identify algebra with coalgebra by means of the inner product (Killing form)

$$g = -\operatorname{Tr}(\mathbf{X}\cdot\mathbf{Y}), \qquad \mathbf{X},\, \mathbf{Y} \in \mathcal{L}. \tag{30}$$

Let's denote coordinates in coalgebra as m_1, m_2, m_3, m_4, p_1, p_2, p_3, p_4, then after the identification we obtain a matrix (an element of algebra):

$$\mathbf{X} = \begin{pmatrix} -i(m_3+m_4) & im_1 + m_2 & \frac{1}{x}(p_1 - ip_2) \\ im_1 - m_2 & i(m_3 - m_4) & \frac{1}{x}(p_3 - ip_4) \\ -p_1 - ip_2 & -p_3 - ip_4 & 2im_4 \end{pmatrix}.$$

The corresponding Lie–Poisson bracket, more precisely, the bundle of brackets linearly dependent on parameter x has the following form for the

coordinate functions of coalgebra

$$\begin{aligned}
\{m_i,\, m_j\} &= \varepsilon_{ijk} m_k, & \{m_i,\, m_4\} &= 0, \\
\{m_i,\, p_j\} &= \frac{1}{2}(\varepsilon_{ijk} p_k - \delta_{ij} p_4), & \{m_i,\, p_4\} &= \frac{1}{2} p_i, \qquad i,\, j,\, k = 1,\, 2,\, 3 \\
\{p_i,\, p_j\} &= \frac{1}{2} x(\varepsilon_{ijk} m_k + 3\varepsilon_{ij3} m_4), & \{p_i,\, p_4\} &= -\frac{1}{2} x(m_i - 3\delta_{i3} m_4).
\end{aligned} \tag{31}$$

In correspondence with the general method developed in [5, 4], we shall construct $\mathbf{L}$ matrix, and its invariants define a commutative set for the whole family of brackets $\{\ \}_\theta + \lambda(\{\ \}_\lambda + \{\ \}_a)$, where $a \in V$ is a shift of the argument. The bracket $\{\}_\theta$ is different from (31); in this bracket variables p_i are pairwise commuting. Let's assume $x = 1$ restricting ourselves to the problem with a real dynamical sense. Then we can present $\mathbf{L}$-matrix as:

$$\mathbf{L} = (\mathbf{h}\lambda + \mathbf{v} + \mathbf{a}\lambda^2),$$

where

$$\begin{aligned}
\mathbf{h} &= \left(\begin{array}{cc|c} -i(m_3+m_4) & im_1+m_2 & \\ im_1-m_2 & i(m_3-m_4) & \smash{\raisebox{1.2ex}{0}} \\ \hline \multicolumn{2}{c|}{0} & 2im_4 \end{array}\right) \\
\mathbf{v} &= \left(\begin{array}{cc|c} & & p_1 - ip_2 \\ \multicolumn{2}{c|}{0} & p_3 - ip_4 \\ \hline -p_1-ip_2 & -p_3-ip_4 & 0 \end{array}\right) \\
\mathbf{a} &= \left(\begin{array}{cc|c} & & a_1 - ia_2 \\ \multicolumn{2}{c|}{0} & a_3 - ia_4 \\ \hline -a_1-ia_2 & -a_3-ia_4 & 0 \end{array}\right).
\end{aligned} \tag{32}$$

Among invariants of matrix $\mathbf{L}$ under arbitrary shift there is a linear on m_i function of the form

$$F_1 = m_4 a^2 + (\boldsymbol{m},\, \boldsymbol{\gamma}_a), \tag{33}$$

where $a^2 = \sum_{i=1}^{4} a_i^2$, $\boldsymbol{m} = (m_1,\, m_2,\, m_3)$, and the components of vector $\boldsymbol{\gamma}_a$ are the functions of a_i of the form

$$\boldsymbol{\gamma}_a = (2(a_1a_3 + a_2a_4),\, 2(-a_2a_3 + a_1a_4),\, a_3^2 + a_4^2 - a_1^2 - a_2^2).$$

Let's suppose $a_1 = a_2 = 0$. In this case integral (33) has the form

$$F_1 = m_3 + m_4. \tag{34}$$

The following square-law invariant of matrix $\mathbf{L}$, we will choose as a Hamiltonian

$$F_2 = \mathrm{Tr}(\mathbf{h}^2 + 2\mathbf{va}) = m_1^2 + m_2^2 + m_3^2 + 3m_4^2 + a_4p_4 + a_3p_3. \tag{35}$$

It defines some (formal) integrable Hamiltonian system on the family of brackets $\{\}_\theta + \lambda(\{\}_\lambda + \{\}_a)$. We can easily find matrix $\mathbf{A}$ for this system [5].

The reduced system and nonlinear Poisson structure. In order to proceed from the found formal system to the generalization of Goryachev–Chaplygin case [7], we shall make the reduction using linear integral (34) directly in obtained $\mathbf{L}$ and $\mathbf{A}$ matrices. In the general case this reduction is not Poisson reduction [5]. We shall make the following substitution in matrix $\mathbf{L}$ (30) and in Hamiltonian (35)

$$m_4 = -m_3 + c, \qquad c = \mathrm{const}\,.$$

We obtain $\mathbf{L}$ matrix and Hamiltonian of integrable system on subalgebra m_1, m_2, m_3, p_1, p_2, p_3, p_4 with gyrostat, the gyrostatic moment being equal to c:

$$\mathbf{L} = \begin{pmatrix} -i\lambda c & (im_1{+}m_2)\lambda & p_1{-}ip_2 \\ (im_1{-}m_2)\lambda & i\lambda(2m_3{-}c) & p_3{-}ip_4{+}(a_3{+}ia_4)\lambda^2 \\ -p_1{-}ip_2 & -p_3{-}ip_4{-}(a_3{+}a_4)\lambda^2 & -2i(m_3{-}c)\lambda \end{pmatrix} \tag{36}$$

$$H = m_1^2 + m_2 + 4m_3^2 - 6m_3c + 2a_4p_4 + 2a_3p_3 \tag{37}$$

$$\mathbf{A} = dH = \begin{pmatrix} i(4m_3 - 3c) & -im_1 - m_2 & 0 \\ -im_1 + m_2 & -i(4m_3 - 3c) & -a_3 + ia_4 \\ 0 & a_3 + ia_4 & 0 \end{pmatrix}.$$

Let's consider Poisson structure, defined by a bracket $\{\}_\theta$. The corresponding Hamiltonian system with Hamiltonian (37) has another linear integral

$$F_3 = m_3 - (\boldsymbol{m}, \boldsymbol{\gamma}) \tag{38}$$

$$\boldsymbol{\gamma} = (2(p_2p_4 + p_1p_3),\, 2(p_2p_3 - p_1p_4),\, p_3^2 + p_4^2 - p_1^2 - p_2^2)),$$

using this integral we can carry out a usual procedure of order reduction (such as Routh reduction or reduction on moment). We can carry out the reduction in the simpler way using algebraic form, if we choose as new variables the integrals of vector field $\boldsymbol{v} = \{\cdot, F_3\}$ of the form [5]

$$\begin{gathered} K_1 = \frac{M_1p_1 + M_2p_2}{\sqrt{p_1^2+p_2^2}}, \quad K_2 = \frac{M_2p_1 - M_1p_2}{\sqrt{p_1^2+p_2^2}}, \quad K_3 = M_3, \\ s_1 = p_3, \quad s_2 = p_4, \quad s_3 = \pm\sqrt{p_1^2+p_2^2}, \end{gathered} \tag{39}$$

having the nonlinear commutation. We describe the transformation (39) in [6]

$$\begin{gathered} \{K_i,\, K_j\} = \varepsilon_{ijk}K_k + \varepsilon_{ij3}\frac{F_4}{s_3^2} \\ \{K_i,\, s_j\} = \varepsilon_{ijk}s_k, \quad \{s_i,\, s_j\} = 0, \end{gathered} \tag{40}$$

where $F_3 = (\boldsymbol{K},\, \boldsymbol{s})s_3$ is a Casimir function of structure (40).

At zero value of "the area integral", i. e. for $F_3 = (\boldsymbol{K},\, \boldsymbol{s})s_3 = 0$ bracket (40) coincides with a usual LiePoisson bracket, defined by algebra $e(3) = so(3) \oplus_s \mathbb{R}^3$, and Hamiltonian (37) in variables (39) coincides with the Hamiltonian of the classical Goryachev–Chaplygin case [7]

$$H^* = \frac{1}{2}H = \frac{1}{2}(K_1^2 + K_2^2 + 4K_3^2) - a_4s_2 - a_3s_1, \quad a_3,\, a_4 = \mathrm{const}\,. \tag{41}$$

Goryachev case with a singular term. We can remove the nonlinearity of structure (40), appearing for $F_3 \neq 0$ on the fixed level $F_3 = c$ using the transformation

$$\boldsymbol{L} = \boldsymbol{K} - c\frac{\boldsymbol{s}}{s_3}.$$

After the transformation bracket (40) gets the form of algebra $e(3)$, and Hamiltonian H^* gets the form

$$H^* = \frac{1}{2}(L_1^2 + L_2^2 + 4L_3^2) - a_4s_2 - a_3s_1 + 3cL_3 + \frac{1}{2}\frac{c^2}{s_3^2}. \tag{42}$$

Hamiltonian (42) can be interpreted as some generalization of the Goryachev–Chaplygin case, at $(\boldsymbol{L},\, \boldsymbol{s}) = 0$, with the additional terms, linear on L_3 and corresponding to the gyrostatic moment. The integrable generalization with

the gyrostatic moment only was described by L. N. Sretensky in [9], the generalization with singular potential only was presented by D. N. Goryachev himself [7], the general case, when it is possible to add to the Hamiltonian both terms with arbitrary independent coefficients, was described in paper [8].

Thus, the presented **L-A** pair is valid for the generalizations of the Goryachev–Chaplygin case. It is different from the mysterious **L-A** pair described in paper [3] which is obtained by deletion of a row and column from the relevant pair of the Kovalevskaya case.

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